The writhe of open and closed curves

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Abstract
Twist and writhe measure basic geometric properties of a ribbon or tube. While
these measures have applications in molecular biology, materials science, fluid
mechanics and astrophysics, they are under-utilized because they are often
considered difficult to compute. In addition, many applications involve curves
with endpoints (open curves); but for these curves the definition of writhe
can be ambiguous. This paper provides simple expressions for the writhe of
closed curves, and provides a new definition of writhe for open curves. The
open curve definition is especially appropriate when the curve is anchored at
endpoints on a plane or stretches between two parallel planes. This definition
can be especially useful for magnetic flux tubes in the solar atmosphere, and
for isotropic rods with ends fixed to a plane.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The writhe of a curve measures how much it kinks and coils. If we add a second curve nearly
parallel to the first, we can further measure how much the second curve twists about the first.
The two curves might be the two sides of a ribbon, or the two strands of a DNA molecule,
or two field lines within a bundle of magnetic flux. When the two curves are both closed
(no endpoints) then we can also measure their linking number. Călugăreanu (1959) proved
a remarkable theorem showing that twist plus writhe equals linking number. The magnetic
analogue states that the total magnetic helicity of a flux tube can be subdivided into twist
helicity and writhe helicity (Berger and Field 1984, Moffatt and Ricca 1992).

Each of the three quantities in the Călugăreanu theorem has special properties. Linking
number is the only topological invariant: deformations of the two curves which do not let
them pass through each other do not change the linking. If we call one curve the axis and
the other the secondary curve, then writhe only depends on the axis curve (unlike linking and
twist). Finally, both linking number and writhe are double integrals, whereas twist is the single
integral of a well-defined density (but, as we will show in this paper, linking and writhe can
also be defined as single integrals in a fixed spatial direction).

The simplest definitions of twist and writhe apply to curves which close upon themselves. This is fine for closed DNA molecules (such as plasmids), but many applications involve curves with endpoints (for example, human DNA, most polymer chains and magnetic fields with endpoints on a boundary surface). One approach for an open DNA molecule involves adding a straight line segment or planar curve to connect the endpoints, then using the closed formulae to calculate writhe or linking (Fuller 1978, Vologodskii and Marko 1997, van der Heijden *et al* 2004). Open writhe can also be defined by closing the associated tantrix curve (see section 6) (Starostin 2005). This paper gives a different method for calculating open writhe, especially suitable for curves or field lines with endpoints on a plane or sphere.

In particular, we show that linking, twist and writhe can be calculated using a single integral in one constant direction in space (e.g. height z). The price we pay is that the curve must be divided into pieces at turning points in z. The integral includes nonlocal terms describing interactions between the pieces, as well as local terms for each piece.

Section 2 reviews the basic ideas behind linking, twist and writhe, following the notation and description in Aldinger *et al* (1995). This section also reviews the idea of a twisted tube, or isotropic rod (van der Heijden and Thompson 2000), which encloses a curve like a cable encloses the wire inside. In some respects tubes are simpler objects than ribbons as they are more symmetric.

Section 3 reviews how linking number can be calculated using a single integral in z. This procedure is a simple example of using Kontsevich integrals (Kontsevich 1993, Chmutov and Duzhin 2000, Berger 2001).

Section 4 applies similar techniques to the closed writhe calculation. Theorem 3 gives the central theorem of this paper: that the rate of growth of the writhe of a curve in the z direction is well defined, in the sense that it is independent of how its surrounding tube is twisted. Theorem 4 gathers together the expressions for writhe in terms of z integrals.

Section 5 defines an open writhe quantity we will call the polar writhe. The polar writhe measures the writhe of a curve between two parallel planes, using the expressions found in section 4.

Section 6 relates the results of the previous sections to a geometric construction called the tantrix curve. It also discusses the results in the context of a theorem of Fuller (1978), which compares the writhe of two curves, one of which can be deformed into the other.

Section 7 gives two examples of writhe computations: first for a twisted parabola with both endpoints on one plane, and secondly for an inclined helical curve.

The more lengthy theorems are placed in an appendix.

2. Definitions and notation

2.1. Ribbons

We will be primarily interested in the geometry of a (non-self-intersecting) curve **x**. As this curve will often be surrounded by other almost parallel curves, we will call it the *axis curve*. Let positions on the axis curve **x** be given by $\mathbf{x}(t)$, for some parametrization t. The most natural parametrization (but not always the easiest to calculate with) has $t = s$, where s measures arclength from some arbitrary starting point. We assume $\mathbf{x}(s)$ is smooth (we will need at least two derivatives). The tangent vector to the axis curve is

$$\widehat{\mathbf{T}}(s) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}. \tag{1}$$

A tangent vector to a curve parametrized by arclength has unit norm, so $|\widehat{\mathbf{T}}(s)| = 1$.
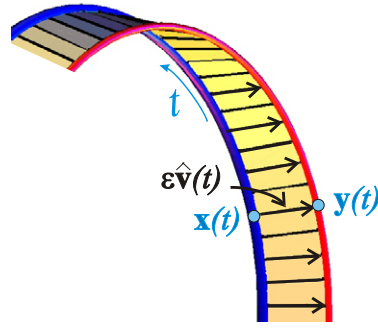
Figure 1. An *axis curve* $\mathbf{x}(t)$, together with a *secondary curve* $\mathbf{y}(t)$.

We now surround our axis curve with structure. To form a ribbon (or a DNA molecule) we will need a secondary curve $\mathbf{y}$ (see figure 1). First, let $\widehat{\mathbf{V}}(t)$ be a unit vector normal to $\widehat{\mathbf{T}}(t)$, i.e. $\widehat{\mathbf{V}}(t) \cdot \widehat{\mathbf{T}}(t) = 0$. This vector points to $\mathbf{y}$ at the point $\mathbf{y}(t) = \mathbf{x}(t) + \epsilon \widehat{\mathbf{V}}(t)$. We generally set $\epsilon \ll 1$ to keep the curves close together. Note that, while we can parametrize both curves by $t = s$, the parameter s measures arclength along the axis curve, not along the secondary curve.

2.2. Tubes

Suppose we draw a circle of radius ϵ centred on $\mathbf{x}(t)$, and perpendicular to $\widehat{\mathbf{T}}(t)$. If we now join up the circles for all t, we obtain a tube enclosing the axis curve. This tube will not intersect itself if we choose ϵ small enough. Suppose the tube contains a secondary curve $\mathbf{y}$. Then the tube surface can be given coordinates (t, ϕ) where the secondary curve defines $\phi = 0$, i.e., $\mathbf{y}(t)$ has coordinates $(t, 0)$. A particular choice of $\mathbf{y}$ and hence a particular coordinate system is called a *framing*. More precisely, if we let $\widehat{\mathbf{W}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{V}}$, then the surface point $\mathbf{y}(t, \phi)$ is given by

$$\mathbf{y}(t, \phi) = \mathbf{x}(t) + \epsilon(\cos \phi \widehat{\mathbf{V}}(t) + \sin \phi \widehat{\mathbf{W}}(t)). \tag{2}$$

In fact, we can fill the tube surface with parallel curves. Given a constant angle $\phi = \beta$ we can define a curve parallel to $\mathbf{y}(t) = \mathbf{y}(t, 0)$ passing through the points $\mathbf{y}(t, \beta)$. We define a *twisted tube* as a tube covered by a family of parallel curves for $0 \leqslant \beta < 2\pi$ (see figure 2). A twisted tube with elastic energy is called an *isotropic rod* (van der Heijden and Thompson 2000).

2.3. The Frenet frame

The local geometry of $\mathbf{x}$ provides an intrinsic set of basis vectors and coordinates, called the *Frenet frame*. Let
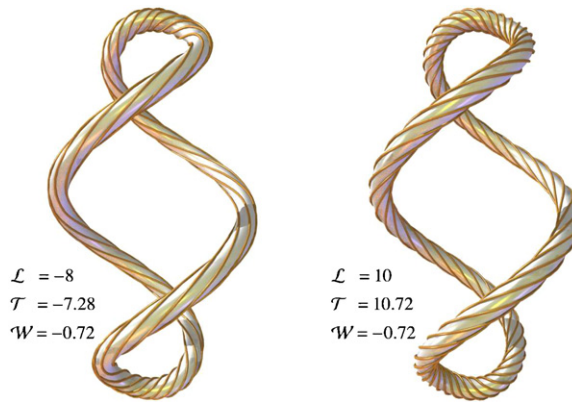
$$\kappa = \left| \frac{\mathrm{d}\widehat{\mathbf{T}}(s)}{\mathrm{d}s} \right| \tag{3}$$

be the *curvature* of $\mathbf{x}$ at s. The *normal* vector is defined (where $\kappa \neq 0$) as

$$\widehat{\mathbf{N}} = \frac{1}{\kappa} \frac{\mathrm{d}\widehat{\mathbf{T}}(s)}{\mathrm{d}s}. \tag{4}$$

As $\widehat{\mathbf{T}}(s)$ is always a unit vector, $\widehat{\mathbf{N}}(s) \cdot \widehat{\mathbf{T}}(s) = 0$. We can now define a third vector, the *binormal*, as

$$\widehat{\mathbf{B}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}}. \tag{5}$$

Figure 2. Two *twisted tubes* with the same axis curves but with differing twist functions (*framings*).

The three vectors $\{\widehat{\mathbf{T}}, \widehat{\mathbf{N}}, \widehat{\mathbf{B}}\}$ form a right-handed orthonormal basis, and satisfy the Frenet–Serret equations

$$\frac{\mathrm{d}\widehat{\mathbf{T}}(s)}{\mathrm{d}s} = \kappa\widehat{\mathbf{N}}, \tag{6}$$

$$\frac{\mathrm{d}\widehat{\mathbf{N}}(s)}{\mathrm{d}s} = \tau\widehat{\mathbf{B}} - \kappa\widehat{\mathbf{T}}, \tag{7}$$

$$\frac{\mathrm{d}\widehat{\mathbf{B}}(s)}{\mathrm{d}s} = -\tau\widehat{\mathbf{T}}, \tag{8}$$

where τ is the *torsion*. Outside of critical points where $\kappa = 0$, we can choose $\widehat{\mathbf{V}}(s) = \widehat{\mathbf{N}}(s)$ to give us a framing for a tube surrounding the axis curve.

2.4. Review of linking number, twist and writhe for closed curves

2.4.1. Crossing number. Suppose two curves are projected onto a plane whose normal points in the direction $\hat{\mathbf{n}}$. We can also regard a projection angle as a viewing angle, i.e., $\hat{\mathbf{n}}$ gives the direction to a distant observer. In the projection plane the curves will cross each other some number of times. Let $C(\hat{\mathbf{n}})$ count the number of positive crossings minus the number of negative crossings. For two distinct closed curves $C(\hat{\mathbf{n}})$ is independent of $\hat{\mathbf{n}}$. Counting crossings can be a convenient method of calculating linking number and writhe (Orlandini *et al* 1994).

2.4.2. Linking number. The Gauß linking number is defined as a double integral over $\mathbf{x}$ (with points labelled $\mathbf{x}(s)$ and tangent $\mathbf{T}_{\mathbf{x}}(s)$) and $\mathbf{y}$ (with points labelled $\mathbf{y}(s')$ and tangent $\mathbf{T}_{\mathbf{y}}(s')$):

$$\mathcal{L}_k \equiv \frac{1}{4\pi} \oint_{\mathbf{x}} \oint_{\mathbf{y}} \mathbf{T}_{\mathbf{x}}(s) \times \mathbf{T}_{\mathbf{y}}(s') \cdot \frac{\mathbf{x}(s) - \mathbf{y}(s')}{|\mathbf{x}(s) - \mathbf{y}(s')|^3} \, \mathrm{d}s \, \mathrm{d}s'. \tag{9}$$

 (i) $\mathcal{L}_k$ is invariant to deformations of the two curves as long as the two curves are not allowed to cross through each other.
(ii) $\mathcal{L}_k$ is an integer.

(iii) $\mathcal{L}_k$ equals half the signed number of crossings of the two curves as seen in any plane projection:

$$\mathcal{L}_k = \tfrac{1}{2} C(\hat{\mathbf{n}}). \tag{10}$$

(iv) $\mathcal{L}_k$ is independent of the direction of the axis curve, i.e. $\mathcal{L}_k$ does not change if $s \to -s$. (For two arbitrary curves, $\mathcal{L}_k$ changes sign if one of the two curves reverses its direction. However, for ribbons both curves must change direction together.)

2.4.3. Writhe

$$\mathcal{W}_r \equiv \frac{1}{4\pi} \oint_{\mathbf{x}} \oint_{\mathbf{x}} \widehat{\mathbf{T}}(s) \times \widehat{\mathbf{T}}(s') \cdot \frac{\mathbf{x}(s) - \mathbf{x}(s')}{|\mathbf{x}(s) - \mathbf{x}(s')|^3} \, \mathrm{d}s \, \mathrm{d}s'. \tag{11}$$

 (i) $\mathcal{W}_r$ depends only on the axis curve $\mathbf{x}$.
(ii) $\mathcal{W}_r$ equals the signed number of crossings of the axis curve with itself, averaged over all possible projection angles (i.e. over all directions on the sphere S^2).
(iii) $\mathcal{W}_r$ is independent of the direction of the axis curve.

2.4.4. Twist

$$\mathcal{T}_w \equiv \frac{1}{2\pi} \oint_{\mathbf{x}} \widehat{\mathbf{T}}(s) \cdot \widehat{\mathbf{V}}(s) \times \frac{\mathrm{d}\widehat{\mathbf{V}}(s)}{\mathrm{d}s} \, \mathrm{d}s \tag{12}$$

$$= \frac{1}{2\pi} \oint_{\mathbf{x}} \frac{1}{|\mathbf{v}|^2(s)} \widehat{\mathbf{T}}(s) \cdot \mathbf{v}(s) \times \frac{\mathrm{d}\mathbf{v}(s)}{\mathrm{d}s} \, \mathrm{d}s \tag{13}$$

where $\mathbf{v} = \epsilon \widehat{\mathbf{V}}$.

 (i) $\mathcal{T}_w$ has a local density along the curve, i.e. it is meaningful to write

$$\mathcal{T}_w = \frac{1}{2\pi} \oint_{\mathbf{x}} \frac{\mathrm{d}\mathcal{T}_w}{\mathrm{d}s} \, \mathrm{d}s. \tag{14}$$

(ii) $\mathrm{d}\mathcal{T}_w/\mathrm{d}s$ measures the rotation rate of the secondary curve about the axis curve. At each point on the axis curve $\mathbf{x}$, define the plane perpendicular to $\mathbf{T}_{\mathbf{x}}(s)$. The offset vector $\widehat{\mathbf{V}}(s)$ lives in this plane, and rotates at a rate

$$\widehat{\mathbf{T}}(s) \cdot \widehat{\mathbf{V}}(s) \times \frac{\mathrm{d}\widehat{\mathbf{V}}(s)}{\mathrm{d}s} = 2\pi \frac{\mathrm{d}\mathcal{T}_w}{\mathrm{d}s}. \tag{15}$$

(iii) For two neighbouring magnetic field lines, the twist is related to the parallel electric current. Let $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ be the electric current associated with the magnetic field $\mathbf{B}$, and $J_\parallel = \mathbf{J} \cdot \mathbf{B}/|\mathbf{B}|$. Then

$$\frac{\mathrm{d}\mathcal{T}_w}{\mathrm{d}s} = \frac{\mu_0 J_\parallel}{4\pi |\mathbf{B}|}. \tag{16}$$

Similarly, if we measure how much two neighbouring flow lines in a fluid twist about each other, then

$$\frac{\mathrm{d}\mathcal{T}_w}{\mathrm{d}s} = \frac{\omega_\parallel}{4\pi |\mathbf{V}|}, \tag{17}$$

where $\mathbf{V}$ is the fluid velocity and ω is the vorticity.

(iv) $\mathcal{T}_w$ is independent of the direction of the axis curve. For example, suppose the axis is a vertical straight line, and the secondary is a right helix (positive twist). Turning the two upside down will still give a right helix of the same pitch.
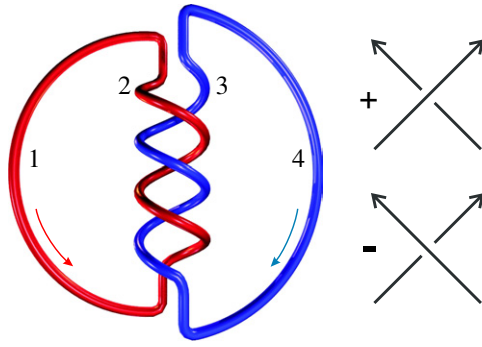
Figure 3. Two closed curves with four crossings. All four are negative (examples of negative and positive crossings are shown to the right of the curves). The linking number equals half the number of signed crossings, i.e. $\mathcal{L}_k = -2$. Alternatively, this linking number can be calculated by adding up the net winding angles $\Delta\Theta_{ij}$ between pieces of the curves (and dividing by 2π). Here $\Delta\Theta_{13} = \Delta\Theta_{14} = \Delta\Theta_{24} = 0$, while $\Delta\Theta_{23} = -4\pi$.

2.5. The Călugăreanu theorem

$$\mathcal{L}_k = \mathcal{T}_w + \mathcal{W}_r. \tag{18}$$

Note that the writhe (11) looks like the linking number (9) applied with both integrals along the axis curve. The Călugăreanu formula can be derived by calculating the linking number in the limit that the ribbon width ϵ shrinks to zero. In this limit, a singularity appears in the integrand along the set of points ('the diagonal') $s = s'$. This is a modest singularity which, when integrated over a small neighbourhood of the diagonal, gives the twist $\mathcal{T}_w$.

3. Linking number in terms of winding numbers in the z direction

The Gauß formula equation (9) gives linking number as a double integral. Sometimes it can be easier to calculate a sum of single integrals along a preferred direction (say along the vertical z axis). We note that the Kontsevich integral for Vassiliev invariants in knot theory (Kontsevich 1993) can be calculated using iterated integrals in one direction (see Berger 2001 for an elementary introduction, and Chmutov and Duzhin 2000 for a thorough treatment). In this section we review the derivation of a directional expression for the linking integral. This will lead in the next section to a directional expression for the writhing number for closed curves. In many cases these integrals may be particularly easy to calculate. Furthermore, these integrals will provide natural definitions for the linking, twist and writhing numbers of open curves with endpoints on a boundary plane or on two parallel boundary planes.

The key to these expressions is the notion of rotation, or winding, about a fixed direction (which we will take to be the z axis). Suppose we scan how two curves evolve in the z direction. Different parts of the curves wind about each other as we scan upwards (see figure 3). Measuring the net *winding angle* will lead us to the linking number. Note that as $\mathcal{L}_k$ is invariant to rotations, a different choice of fixed direction will give the same linking number.

Consider two curves $\mathbf{x}_i$ and $\mathbf{x}_j$. For the moment suppose that these curves always travel upwards in z, i.e., along each curve $dz/ds > 0$ where s is arclength. We can then use z to parametrize the curves, and label points as $\mathbf{x}_i(z)$ or $\mathbf{x}_j(z)$. Let the relative position vector at height z be $\mathbf{r}_{ij}(z) = \mathbf{x}_j(z) - \mathbf{x}_i(z)$. Note that $\mathbf{r}_{ij}(z)$ is parallel to the xy plane. We will let

$\Theta_{ij}(z)$ be the orientation of this vector with respect to the x axis; as we travel upwards in z, $\mathbf{r}_{ij}(z)$ may rotate, changing $\Theta_{ij}(z)$. This rotation of the relative position vector about the z axis will give us a simple method of calculating linking and writhe. Let $\hat{\mathbf{z}}$ be the unit vector in the z direction. The rotation rate is given by

$$\frac{\mathrm{d}\Theta_{ij}}{\mathrm{d}z} = \frac{\hat{\mathbf{z}} \cdot \mathbf{r}_{ij}(z) \times \mathbf{r}'_{ij}(z)}{|\mathbf{r}_{ij}(z)|^2}. \tag{19}$$

The net winding angle (which can be more than 2π) between two heights z_1 and z_2 is simply the integral of this rate:

$$\Delta\Theta_{ij} = \int_{z_1}^{z_2} \frac{\mathrm{d}\Theta_{ij}}{\mathrm{d}z}. \tag{20}$$

We note that $\mathcal{T}_w$ arises from rotation of the secondary curve about a constantly varying axis direction. Here we are expressing $\mathcal{L}_k$ and $\mathcal{W}_r$ in terms of rotation about the (non-varying) z direction.

Now let us drop the assumption that the two curves always travel upwards. A generic curve (of finite length) has a finite set of discrete points where $\mathrm{d}s/\mathrm{d}z = 0$. These points divide the curve into pieces where $\mathrm{d}s/\mathrm{d}z > 0$ or $\mathrm{d}z/\mathrm{d}s < 0$. Say curve $\mathbf{x}$ has n of these pieces, and label them with $i = 1, \ldots, n$. Piece i starts at some height z_i and ends at height z_{i+1}. Let $\sigma_i(z)$ tell us whether piece i exists at height z, and whether it is rising or falling:

$$\sigma_i(z) = \begin{cases} 1, & z \in (z_i, z_{i+1}) \text{ and } \mathrm{d}s/\mathrm{d}z > 0, \\ -1, & z \in (z_i, z_{i+1}) \text{ and } \mathrm{d}s/\mathrm{d}z < 0, \\ 0, & z \notin (z_i, z_{i+1}). \end{cases} \tag{21}$$

Theorem 1. *Let curve $\mathbf{x}$ have pieces $i = 1, \ldots, n$ and curve $\mathbf{y}$ have pieces $j = 1, \ldots, m$. Let Θ_{ij} be the orientation of the relative vector $\mathbf{r}_{ij} = \mathbf{y}_j(z) - \mathbf{x}_i(z)$. The linking number between the two curves is given by*

$$\mathcal{L}_k = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_i \sigma_j \frac{\mathrm{d}\Theta_{ij}}{\mathrm{d}z} \, \mathrm{d}z \tag{22}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\sigma_i \sigma_j}{2\pi} \Delta\Theta_{ij}. \tag{23}$$

The proof will be given in the appendix. This theorem gives the linking number as a sum of single integrals, or alternatively in terms of a sum of winding angles $\Delta\Theta_{ij}$.

Let $z^{\min}$ and $z^{\max}$ be the minimum and maximum heights which both curves reach, and let $z^{\min} < z_0 < z^{\max}$. If we now define

$$\widetilde{\mathcal{L}}(z_0) = \int_{-\infty}^{z_0} \frac{\mathrm{d}\widetilde{\mathcal{L}}}{\mathrm{d}z} \, \mathrm{d}z = \int_{z^{\min}}^{z_0} \frac{\mathrm{d}\widetilde{\mathcal{L}}}{\mathrm{d}z} \, \mathrm{d}z, \qquad \text{where} \tag{24}$$

$$\frac{\mathrm{d}\widetilde{\mathcal{L}}}{\mathrm{d}z} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\mathrm{d}\widetilde{\mathcal{L}}_{ij}}{\mathrm{d}z}, \qquad \frac{\mathrm{d}\widetilde{\mathcal{L}}_{ij}}{\mathrm{d}z} = \frac{\sigma_i \sigma_j}{2\pi} \frac{\mathrm{d}\Theta_{ij}}{\mathrm{d}z}, \tag{25}$$

then $\widetilde{\mathcal{L}}(z_0)$ gives the net winding of the curves below z_0, and $\widetilde{\mathcal{L}}(z^{\max}) = \mathcal{L}_k$. Note that at a given height z_0, $\widetilde{\mathcal{L}}(z_0)$ will not be a topological invariant to arbitrary motions of the two

curves. However, it does measure something about how much the two curves are entwined in the half-space below z_0. There remains a restricted sense of topological invariance:

Theorem 2. *The net winding number* $\widetilde{\mathcal{L}}(z_0)$ *is an invariant to the restricted set of motions which vanish at* $z = z_0$ *(such motions do not move the intersection points of the curves with* $z = z_0$, *nor do they allow other parts of the curves to pass through this plane).*

Proof of theorem 2. The total linking number $\mathcal{L}_k$ is invariant to all motions (as long as the two curves do not pass through each other). Consider the net winding of the curves above z_0,

$$\widetilde{\mathcal{L}}(z_0, z_{\max}) \equiv \mathcal{L}_k - \widetilde{\mathcal{L}}(z_0) = \int_{z_0}^{z_{\max}} \frac{d\widetilde{\mathcal{L}}}{dz}\, dz. \tag{26}$$

First consider the restricted set of motions vanishing *at and above* z_0. As the curves do not change their shape above z_0, the net winding above this plane $\widetilde{\mathcal{L}}(z_0, z_{\max})$ does not change. Since $\mathcal{L}_k$ is invariant, their difference $\widetilde{\mathcal{L}}([z_{\min}, z_0]) = \widetilde{\mathcal{L}}(z_0) = \mathcal{L}_k - \widetilde{\mathcal{L}}([z_0, z_{\max}])$ does not change either. Thus motions *below* z_0 do not affect the net winding of the curves below.

Now suppose there are motions both above and below z_0, but the boundary plane $z = z_0$ remains frozen. The extra motions above z_0 do not change the shape of the curves below, so the conclusions of the previous paragraph still hold.

4. The writhe of closed tubes

4.1. Decomposition into local and nonlocal parts

We can handle the writhe of a closed tube in a similar manner to the linking number. We employ the techniques from the previous section to calculate $\mathcal{L}_k$, then subtract $\mathcal{T}_w$. As mentioned in section 2.4.3, for a whole curve $\mathcal{W}_r = \mathcal{L}_k - \mathcal{T}_w$ only depends on the axis curve. For example, figure 2 shows two twisted tubes with the same axis curve. As they have different families of secondaries (i.e. different framings), they have differing values of $\mathcal{L}_k$ and $\mathcal{T}_w$, but the same value of $\mathcal{W}_r$.

Suppose we now define $\widetilde{\mathcal{T}}(z_0)$ in the same way as $\widetilde{\mathcal{L}}(z_0)$. This will give the net twist of the section or sections of a curve lying below the plane $z = z_0$. We would then wish to define a new quantity $\widetilde{\mathcal{W}}(z_0) \equiv \widetilde{\mathcal{L}}(z_0) - \widetilde{\mathcal{T}}(z_0)$. But does this new quantity depend only on the axis curve? If so, then we have some justification for interpreting it as the part of the total writhe arising from the geometry of the axis curve below z_0.

Unfortunately, for ribbons $\widetilde{\mathcal{L}}(z_0) - \widetilde{\mathcal{T}}(z_0)$ depends on the shape of the secondary as well as the axis. The good news, however, is that for twisted tubes the construction works! In particular, theorem 3 below shows that if we average $\widetilde{\mathcal{L}}(z_0) - \widetilde{\mathcal{T}}(z_0)$ over the family of secondaries, the result depends only on the axis curve, as desired. There is no longer any dependence on framing.

We first define

$$\widetilde{\mathcal{W}}(z_0) = \int_{z_{\min}}^{z_0} \frac{d\widetilde{\mathcal{W}}(z)}{dz}\, dz, \tag{27}$$

$$\widetilde{\mathcal{T}}(z_0) = \int_{z_{\min}}^{z_0} \frac{d\widetilde{\mathcal{T}}(z)}{dz}\, dz \tag{28}$$

where $d\widetilde{\mathcal{W}}/dz$ and $d\widetilde{\mathcal{T}}/dz$ remain to be determined. For consistency with the total writhe and twist, we must have $\widetilde{\mathcal{W}}(z^{\max}) = \mathcal{W}_r$ and $\widetilde{\mathcal{T}}(z^{\max}) = \mathcal{T}_w$.

To simplify our expressions, we will denote derivatives with respect to z with a prime. Begin by again considering an axis curve $\mathbf{x}$ with a nearby secondary curve $\mathbf{y} = \mathbf{x} + \epsilon \widehat{\mathbf{V}}$ twisting around it. The curves are divided into pieces $\mathbf{x}_i$ and $\mathbf{y}_i$ at extrema in z. Along each piece we can calculate the twist according to equation (12). Let the contribution from piece i be $\mathcal{T}_{wi}$. Piece i travels from height $z_i^{\min}$ to height $z_i^{\max}$. Recall that σ_i gives the sign of $\mathrm{d}s/\mathrm{d}z$ (so if $\sigma_i = +1$ then $s = s_i^{\min}$ at $z = z_i^{\min}$). Then

$$\mathcal{T}_{wi} = \int_{s_i^{\min}}^{s_i^{\max}} \frac{\mathrm{d}\mathcal{T}_{wi}}{\mathrm{d}s}\,\mathrm{d}s = \int_{z_i^{\min}}^{z_i^{\max}} \frac{\mathrm{d}\mathcal{T}_{wi}}{\mathrm{d}s} \left| \frac{\mathrm{d}s}{\mathrm{d}z} \right|\,\mathrm{d}z, \tag{29}$$

and thus

$$\widetilde{T}_i'(z) = \frac{\mathrm{d}\mathcal{T}_{wi}}{\mathrm{d}s} \left| \frac{\mathrm{d}s}{\mathrm{d}z} \right| = \frac{\sigma_i}{2\pi} \widehat{\mathbf{T}}_i(z) \cdot \widehat{\mathbf{V}}_i(z) \times \widehat{\mathbf{V}}_i'(z). \tag{30}$$

Next consider the net winding number $\widetilde{\mathcal{L}}$. Recall that this sums winding numbers over all pairs of pieces $\mathbf{x}_i$ and $\mathbf{y}_j$. A particular piece of the axis curve $\mathbf{x}_i$ winds locally about its nearby secondary piece $\mathbf{y}_i$, and nonlocally about other far away pieces of the secondary curve $\mathbf{y}_j$, $j \neq i$. Thus we have

$$\widetilde{\mathcal{L}}'(z) = \sum_{i=1}^{n} \widetilde{\mathcal{L}}'(\mathbf{x}_i, \mathbf{y}_i)(z) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \widetilde{\mathcal{L}}'(\mathbf{x}_i, \mathbf{y}_j)(z). \tag{31}$$

For the nonlocal terms, we can substitute $\mathbf{x}_j$ for $\mathbf{y}_j$, since the distance ϵ between the axis and secondary curve is much smaller than the distance between pieces i and j. Thus

$$\widetilde{\mathcal{L}}'(\mathbf{x}_i, \mathbf{y}_j) = \widetilde{\mathcal{L}}'(\mathbf{x}_i, \mathbf{x}_j) \equiv \widetilde{\mathcal{L}}_{ij}'. \tag{32}$$

Letting $\widetilde{\mathcal{L}}_i' \equiv \widetilde{\mathcal{L}}'(\mathbf{x}_i, \mathbf{y}_i)$,

$$\widetilde{\mathcal{L}}'(z) = \sum_{i=1}^{n} \widetilde{\mathcal{L}}_i'(z) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \widetilde{\mathcal{L}}_{ij}'. \tag{33}$$

The double sum includes both (i, j) and (j, i) pairs. This happens because $\mathbf{y}_i$ winds about axis piece $\mathbf{x}_j$, just as $\mathbf{x}_i$ winds about the secondary piece $\mathbf{y}_j$, and both must be included. Note that the nonlocal terms $\widetilde{\mathcal{L}}_{ij}'(z)$ do not depend on the choice of secondary curve $\mathbf{y}$, but the local terms $\widetilde{\mathcal{L}}_i'(z)$ do.

We can decompose twist $\widetilde{T}'$ and writhe $\widetilde{\mathcal{W}}'$ in a similar manner: there is only a local contribution to $\widetilde{T}_i'$ in equation (30), so

$$\widetilde{T}' = \sum_{i=1}^{n} \widetilde{T}_i'(z). \tag{34}$$

Writhe will contain both local and nonlocal terms:

$$\widetilde{\mathcal{W}}' = \sum_{i=1}^{n} \widetilde{\mathcal{W}}_i'(z) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \widetilde{\mathcal{W}}_{ij}'(z), \tag{35}$$

$$\widetilde{\mathcal{W}}_i'(z) = \widetilde{\mathcal{L}}_i'(z) - \widetilde{T}_i'(z), \tag{36}$$

$$\widetilde{\mathcal{W}}_{ij}'(z) = \widetilde{\mathcal{W}}_{ji}'(z) = \widetilde{\mathcal{L}}_{ij}'(z) = \frac{\sigma_i \sigma_j}{2\pi} \Theta_{ij}'(z). \tag{37}$$

In what follows, it will often be convenient to decompose the tangent vector $\widehat{\mathbf{T}}$ into its z component T_z and its perpendicular components $\widehat{\mathbf{T}}_\perp$. Suppose $\widehat{\mathbf{T}}$ is oriented at an angle θ with respect to the z axis. We will define

$$\lambda = \cos\theta = T_z = \mathrm{d}z/\mathrm{d}s, \tag{38}$$

$$\mu = \sin\theta = |\widehat{\mathbf{T}}_\perp|. \tag{39}$$

4.2. The local contribution to the writhe

As mentioned at the beginning of the previous section, we must average the quantity $\widetilde{\mathcal{L}}(z) - \widetilde{\mathcal{T}}(z)$ over all secondary curves in a twisted tube. The following theorem shows that this procedure removes all dependence on framing (i.e. dependence on how the tube is twisted).

Theorem 3. *Let*

$$\widetilde{\mathcal{W}}_i'(z) = \langle \widetilde{\mathcal{L}}_i'(z) - \widetilde{\mathcal{T}}_i'(z) \rangle, \tag{40}$$

where $\langle \rangle$ denotes an average over all secondary curves in the surface of a twisted tube. Then $\widetilde{\mathcal{W}}_i'(z)$ is independent of framing, and is given by

$$\widetilde{\mathcal{W}}_i'(z) = \frac{1}{2\pi} \frac{1}{(1 + |\lambda_i|)} (\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z; \tag{41}$$

in terms of the curvature κ and binormal $\widehat{\mathbf{B}}$,

$$\widetilde{\mathcal{W}}_i'(z) = \frac{1}{2\pi} \frac{1}{(1 + |\lambda_i|)} \frac{\kappa_i B_{zi}}{\lambda_i}. \tag{42}$$

Note that this implies $\widetilde{\mathcal{W}}(z_0)$ only depends on the shape of the section or sections of the axis curve in the half-space below z_0. The proof of framing independence is somewhat detailed, and is left for the appendix. Here we will calculate $\widetilde{\mathcal{W}}_i'(z)$ given a particularly simple choice of framing, where $\widehat{\mathbf{V}}$ is always horizontal.

At each point on the axis, we define three orthonormal vectors $\{\widehat{\mathbf{T}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\}$ starting with the tangent vector, with

$$\widehat{\mathbf{f}} = \frac{\widehat{\mathbf{z}} \times \widehat{\mathbf{T}}}{\mu}; \tag{43}$$

$$\widehat{\mathbf{g}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{f}} \tag{44}$$

(points where $\widehat{\mathbf{T}}$ is parallel to the z axis do not cause any real difficulty, as alternative framings can be employed near them which do not change the final answer). We now choose $\widehat{\mathbf{V}} = \widehat{\mathbf{f}}$ and $\widehat{\mathbf{W}} = \widehat{\mathbf{g}}$.

The twist number expression equation (30) becomes (using $T_z = \mathrm{d}z/\mathrm{d}s = \sigma_i |\lambda_i|$)

$$2\pi \widetilde{\mathcal{T}}_i'(z) = \frac{\sigma_i}{\mu_i^2} \widehat{\mathbf{T}}_i \cdot (\widehat{\mathbf{z}} \times \widehat{\mathbf{T}}_i) \times (\widehat{\mathbf{z}} \times \widehat{\mathbf{T}}_i') \tag{45}$$

$$= \frac{\sigma_i}{\mu_i^2} T_z (\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z \tag{46}$$

$$= \frac{|\lambda_i|}{\mu_i^2} \widehat{\mathbf{z}} \cdot \widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i'. \tag{47}$$

Next, rotation of the axis and the secondary curve about each other along segment i contribute to the linking number. The relative position vector is $\mathbf{y}_i(z) - \mathbf{x}_i(z) = \epsilon \widehat{\mathbf{f}}(z)$, so

$$2\pi \widetilde{\mathcal{L}}_i' = \frac{d\Theta}{dz} = \hat{\mathbf{z}} \cdot \widehat{\mathbf{f}}_i \times \widehat{\mathbf{f}}_i' \tag{48}$$

$$= \frac{1}{\mu_i^2}(\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z. \tag{49}$$

Taking the difference between local linking and twist,

$$2\pi \widetilde{\mathcal{W}}_i' = \frac{1 - |\lambda_i|}{\mu_i^2}(\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z = \frac{1}{1 + |\lambda_i|}(\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z. \tag{50}$$

Now,

$$\widehat{\mathbf{T}}' = \frac{1}{\lambda}\frac{d\widehat{\mathbf{T}}}{ds} = \frac{\kappa}{\lambda}\widehat{\mathbf{N}}, \tag{51}$$

so

$$(\widehat{\mathbf{T}} \times \widehat{\mathbf{T}}')_z = \frac{\kappa}{\lambda}B_z. \tag{52}$$

These results (equations (25), (37) and (41)) can be summarized as follows:

Theorem 4.

$$\mathcal{W}_r = \widetilde{\mathcal{W}}_{\text{local}} + \widetilde{\mathcal{W}}_{\text{nonlocal}}, \tag{53}$$

$$\widetilde{\mathcal{W}}_{\text{local}} = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{z_i^{\min}}^{z_i^{\max}}\frac{1}{(1 + |\lambda_i|)}(\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z \, dz, \tag{54}$$

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n}\frac{\sigma_i\sigma_j}{2\pi}\int_{z_{ij}^{\min}}^{z_{ij}^{\max}}\Theta_{ij}'(z) \, dz. \tag{55}$$

4.3. The nonlocal contribution to the writhe

As an example, we will consider a simple heart shaped curve (see figure 4). The curve divides into four pieces, so there will be six pairs of pieces going into the double sum. First,

$$\widetilde{\mathcal{W}}_{12} = -\frac{1}{2\pi}\int_{z_C}^{z_B}\Theta_{12}'(z) \, dz. \tag{56}$$

Let ϕ_B be the orientation of the tangent vector at B in the xy plane, i.e. $\tan\phi_B = T_{By}/T_{Bx}$. Also let Θ_{PC} be the orientation of the vector pointing from P to C. Then Θ_{12} starts out pointing in the direction ϕ_B and ends pointing in the direction Θ_{PC}. Thus

$$\widetilde{\mathcal{W}}_{12} = \frac{1}{2\pi}(\Theta_{PC} - \phi_B + 2\pi w_{12}) \tag{57}$$

for some integer w_{12} (which keeps track of complete turns).

The remaining possibilities work the same way, except for Θ_{14} which starts out pointing in the *opposite* direction to ϕ_A. Thus

$$\widetilde{\mathcal{W}}_{13} = \frac{1}{2\pi}(\Theta_{BQ} - \Theta_{PC} + 2\pi w_{13}), \tag{58}$$
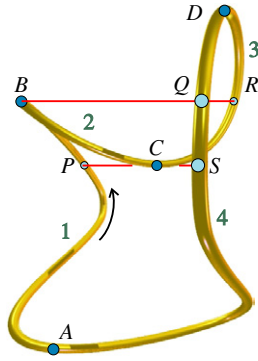
Figure 4. The axis of the tube is a heart-shaped curve with maxima at points B and D, and minima at points A and C. For this example $z_D > z_B > z_C > z_A$. The points C, P and S are at height z_C, while B, Q and R are at height z_B. Piece 1 goes from A to B, piece 2 from B to C, piece 3 from C to D, and piece 4 from D to A.

$$\widetilde{\mathcal{W}}_{14} = \frac{1}{2\pi}((\phi_A \pm \pi) - \Theta_{BR} + 2\pi w_{14}), \tag{59}$$

$$\widetilde{\mathcal{W}}_{23} = \frac{1}{2\pi}(\phi_C - \Theta_{BQ} + 2\pi w_{23}), \tag{60}$$

$$\widetilde{\mathcal{W}}_{24} = \frac{1}{2\pi}(\Theta_{BR} - \Theta_{CS} + 2\pi w_{24}), \tag{61}$$

$$\widetilde{\mathcal{W}}_{34} = \frac{1}{2\pi}(\Theta_{CS} - \phi_D + 2\pi w_{34}). \tag{62}$$

The sum is (remember to include $\widetilde{\mathcal{W}}_{21} = \widetilde{\mathcal{W}}_{12}$, etc)

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \frac{1}{\pi}(\phi_A - \phi_B + \phi_C - \phi_D) + 2w \pm 1, \qquad w = \sum_{i<j} w_{ij}. \tag{63}$$

Thus we can conclude, without calculating the winding numbers w_{ij} (or worrying about the dependence of ϕ_A, ϕ_B, etc, on the position of the branch cut), that

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \frac{1}{\pi}(\phi_A - \phi_B + \phi_C - \phi_D) - 1 \qquad \text{mod } 2. \tag{64}$$

Calculating mod 2 can be useful in several situations: if the writhe is known to be small (i.e. between -1 and 1); if a curve is evolving smoothly from an initial configuration with known writhe, so that the exact value can be calculated by continuity; or if one can calculate the exact value numerically with low resolution, and then increase accuracy by comparing with the mod 2 formula.

The above result generalizes to all closed curves. We will consider a closed curve with n pieces and hence n turning points. Suppose piece i stretches between a minimum at $z_i^{\min}$ and a maximum at $z_i^{\max}$. Note that these points will be labelled twice: for example, if piece i has $ds/dz > 0$ then the point at $z_i^{\max}$ joins piece i with piece $i+1$ and so $z_i^{\max} = z_{i+1}^{\max}$.

Theorem 5. *Let $\phi(z_i^{\min})$ and $\phi(z_i^{\max})$ be the angles with respect to the x axis of the tangent vectors $\widehat{\mathbf{T}}(z_i^{\min})$ and $\widehat{\mathbf{T}}(z_i^{\max})$. Then*

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \frac{1}{2\pi} \sum_{i=1}^{n} \left(\phi(z_i^{\min}) - \phi(z_i^{\max}) \right) \quad - 1 \qquad \text{mod } 2. \tag{65}$$

(Proof in appendix.)

5. The writhe of open curves

5.1. The polar writhe

The previous sections have shown that writhe can be calculated by dividing a closed curve at maxima and minima in some coordinate z, then performing single integrals. The calculations involve the quantities $\widetilde{\mathcal{T}}(z)$, $\widetilde{\mathcal{L}}(z)$ and $\widetilde{\mathcal{W}}(z)$ which integrate from $-\infty$ to z. The quantity $\widetilde{\mathcal{L}}$ measures net winding of two curves about each other relative to the z direction. Note that if we wish to consider the amount of winding between the planes $z = z_0$ and $z = z_1$ we need only calculate $\widetilde{\mathcal{L}}(z_0, z_1) \equiv \widetilde{\mathcal{L}}(z_1) - \widetilde{\mathcal{L}}(z_0)$. By theorem 2, $\widetilde{\mathcal{L}}(z_0)$ has topological meaning: it is invariant to motions of the curve which vanish at the plane $z = z_0$. Twist is a local quantity, so it makes sense to ask how much twist occurs below the plane $z = z_0$. What about writhe?

For twisted tubes it makes sense to define a writhe measure for the half-space below z_0 using $\widetilde{\mathcal{W}}(z_0)$: by theorem 3 this quantity is independent of framing, i.e., it only depends on the axis curve. We can generalize this measure to arbitrary directions. Let us write $\widetilde{\mathcal{W}}(z_0) = \widetilde{\mathcal{W}}(z_0, \hat{z})$ to explicitly show the dependence on the z direction. In the same way, let $\hat{q}$ be some other constant unit vector. Define q to be a Cartesian coordinate increasing in the direction of $\hat{q}$. Then we can define an open writhe $\widetilde{\mathcal{W}}(q_0, \hat{q})$ relative to the $\hat{q}$ direction.

Also we can define

$$\widetilde{\mathcal{W}}(q_0, q_1, \hat{q}) = \widetilde{\mathcal{W}}(q_1, \hat{q}) - \widetilde{\mathcal{W}}(q_0, \hat{q}) \tag{66}$$

to be our writhe measure between the planes $q = q_0$ and $q = q_1$.

What should this writhe measure be called? We would love to use the term 'directional writhe', since $\widetilde{\mathcal{W}}(q_0, \hat{q})$ depends on the direction $\hat{q}$. Unfortunately, this term is already in use. (Briefly, take a closed curve and make an exact duplicate. Then translate the duplicate a small distance in the $\hat{q}$ direction. The linking number between the original curve and its duplicate is the directional writhe with respect to $\hat{q}$. Averaging over all directions gives the writhe (Aldinger *et al* 1995).)

We will instead call $\widetilde{\mathcal{W}}(q_0, \hat{q})$ the *polar writhe with respect to* $\hat{q}$. The word 'polar' relates to the poles of the tantrix sphere described in the next section. For simplicity, we will always choose $\hat{q} = \hat{z}$ and simply call the polar writhe below z_0 by $\widetilde{\mathcal{W}}(z_0)$ and the polar writhe between z_0 and z_1 by $\widetilde{\mathcal{W}}(z_0, z_1)$.

Note that equations (54) and (55) do not involve arclength s except through the quantity $|\lambda| = |dz/ds|$. Thus a reversal of the direction of the axis curve (equivalent to $s \to -s$) does not affect the polar writhe. In addition, while polar writhe in general depends on the special direction $\hat{q}$, a reversal $\hat{q} \to -\hat{q}$ does not change polar writhe; to see this for $\hat{q} = \hat{z}$, let $w = -z$. We need to show that

$$\widetilde{\mathcal{W}}(z_0, z_1, \hat{z}) = \widetilde{\mathcal{W}}(-w_1, -w_0, \hat{w}). \tag{67}$$

To see this, first note that the $\sigma_i \sigma_j$ factor in equation (55) stays the same, as both σ_i and σ_j reverse sign. Also,

$$(\widehat{\mathbf{T}}_i \times \widehat{\mathbf{T}}_i')_z = \widehat{\mathbf{T}}_i \times \frac{d\widehat{\mathbf{T}}_i}{dz} \cdot \hat{z} = \widehat{\mathbf{T}}_i \times \frac{d\widehat{\mathbf{T}}_i}{dw} \cdot \hat{w}. \tag{68}$$

More subtly, the quantity

$$\frac{d\theta}{dz} \to \frac{d\theta}{dw} \tag{69}$$

without change. For example, if we flip over a right helix, it remains a right helix (note that a turn through an angle θ about the z axis is actually a pseudoscalar under the improper transformation $x \to x, y \to y, z \to -z$).
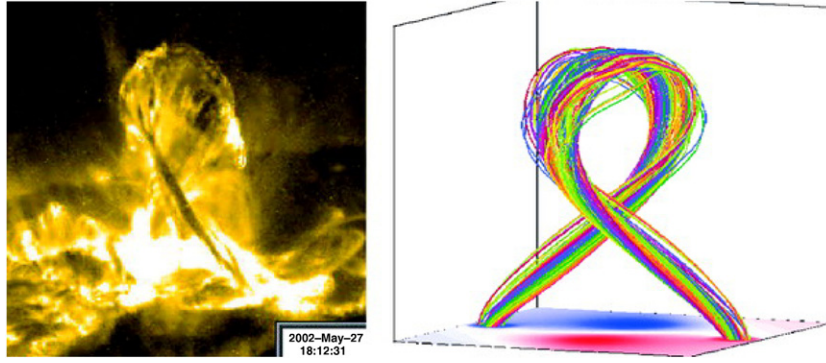
Figure 5. A kinked magnetic filament in the solar corona on 27 May 2002. On the left is an image from the TRACE 195 Å line. The right picture is from a numerical simulation by Török and Kliem 2005.

5.2. Example—simple loops

As an example, consider an open curve with endpoints on the same boundary plane $z = 0$. We will call this a simple loop if there is only one turning point in z (at $z^{\max}$). Such loops are of importance in astrophysics: magnetic loops form in the solar atmosphere (see figure 5), and may occur in the atmospheres of accretion disks. The loop will have two pieces, or legs.

Let the polar writhe of the loop be $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(0, \infty)$. By equations (54), (55), and (65),

$$\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}_1 + \widetilde{\mathcal{W}}_2 + 2\widetilde{\mathcal{W}}_{12}, \tag{70}$$

$$\widetilde{\mathcal{W}}_1 = \frac{1}{2\pi} \int_1 \frac{1}{(1 + |\lambda_1|)} (\widehat{\mathbf{T}}_1 \times \widehat{\mathbf{T}}'_1)_z \, \mathrm{d}z, \tag{71}$$

$$\widetilde{\mathcal{W}}_2 = \frac{1}{2\pi} \int_2 \frac{1}{(1 + |\lambda_2|)} (\widehat{\mathbf{T}}_2 \times \widehat{\mathbf{T}}'_2)_z \, \mathrm{d}z, \tag{72}$$

$$\widetilde{\mathcal{W}}_{12} = \frac{1}{2\pi} (\Theta_{12} - \phi(z^{\max})) + w_{12}. \tag{73}$$

For modest writhing, one should only see at most one crossing of the two legs when viewed from the side. As writhing averages crossing numbers over all projection angles, this would imply $\widetilde{\mathcal{W}}_{12} < 1$ (thus setting w_{12}).

5.3. Polar writhe and magnetic tubes

Linking number can be applied to vector fields as well as curves. Suppose the integral curves (field lines) of a divergence-free vector field are all closed. The *helicity integral* measures the linking number of the field lines, averaged over all pairs of field lines, and weighted by flux (Moffatt 1969). Even when the field lines are ergodic and do not close upon themselves this picture is still valid (Arnold and Khesin 1998). Consider a magnetic field confined to a thin closed tube with net flux Φ. The helicity H of the tube can be decomposed into contributions from twist and writhe, i.e. $H = \mathcal{T}_w \Phi^2 + \mathcal{W}_r \Phi^2$ where $\mathcal{W}_r$ is the writhe of the axis of the tube, and $\mathcal{T}_w$ measures the average twist of other field lines (acting as secondary curves) about the axis (Berger and Field 1984, Moffatt and Ricca 1992).
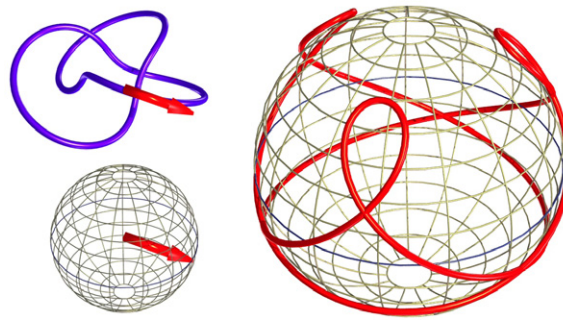
Figure 6. A 2–3 torus (trefoil) knot and its associated tantrix. The upper left figure shows the knot with a tangent vector drawn at one point along the curve. Below, the vector has been drawn so that its tip lies on a unit sphere. The figure to the right displays the full tantrix for the knot (for this curve, $\mathcal{W}_r = 3.52$).

One often needs to calculate the helicity contained within some region of space $\mathcal{V}$. If field lines cross the boundary $\mathcal{S}$ of this region, then they will not be closed within the region. In this case helicity can be measured relative to the minimum-energy vacuum magnetic field (Berger and Field 1984).

Suppose we slice space into a set of layers separated by parallel planar boundaries at $z = z_0, z_1, \dots$. Then the helicity of all space will equal the sum of the helicities of each layer. This situation corresponds to the constructions made in this paper. The helicity of all space, in fact, can be written as an integral in z corresponding to equation (22) averaged over all pairs of field lines. As a consequence, we can still write the helicity content of part of a flux tube, sliced between planes z_i and z_j, as a sum of twist and writhe:

$$H(z_i, z_j) = \widetilde{\mathcal{T}}(z_i, z_j)\Phi^2 + \widetilde{\mathcal{W}}(z_i, z_j)\Phi^2. \qquad (74)$$

6. Writhe and tantrix area

We can increase our geometrical understanding of writhe by constructing *tantrix curves*. The tangent vector $\widehat{\mathbf{T}}$ (as it is by definition a unit vector) maps points on a curve to points on a unit sphere. Call this the *tantrix sphere*, and let the *tantrix* or tantrix curve be the path the tip of the tangent vector takes on this sphere (see figure 6). An important theorem (Fuller 1978, Aldinger *et al* 1995) shows that the writhe of a closed curve is related to the spherical area A enclosed by the tantrix sphere:

$$\mathcal{W}_r = \frac{A}{2\pi} - 1 \qquad \mod 2. \qquad (75)$$

6.1. The local writhe term

Here we show that our expressions for the polar writhe of open curves also bear simple relations to tantrix area. First consider a section of a curve for which $ds/dz > 0$, so that the tantrix stays in the northern hemisphere of the tantrix sphere. For a single section like this, there will only be a local contribution $\widetilde{\mathcal{W}}_{\text{local}}$, as in equations (41) and (54).

Theorem 6. *The polar writhe of a single curve section for which* $ds/dz > 0$ *equals* area$/2\pi$ *between the tantrix and the north pole (where the ends of the tantrix are joined to the north*

pole by geodesics). This area is defined to be positive if the tantrix winds about the pole in a right-handed sense.

Similarly, the polar writhe of a single curve section for which $ds/dz < 0$ *equals* area/2π *between the tantrix and the south pole.*

This theorem gives an exact number, not modulo 2. Note that a closed curve drawn on a sphere divides the surface into two pieces; the curve winds about one piece in a right-handed sense and the other in a left-handed sense (someone travelling due east on a latitude line will sweep out a positive area between the latitude line and the north pole; but also a negative area between the latitude line and the south pole).

Proof of theorem 6. Recall from equation (38) that $T_z = \lambda = \cos\theta$; thus θ gives the co-latitude on the tantrix sphere. Now

$$\widehat{\mathbf{T}}' \, dz = d\widehat{\mathbf{T}} = d\theta \, \hat{\theta} + \sin\theta \, d\phi \, \hat{\phi}. \tag{76}$$

Also $\hat{z} \times \widehat{\mathbf{T}} = \sin\theta\hat{\phi}$, so

$$(\widehat{\mathbf{T}} \times \widehat{\mathbf{T}}') \cdot \hat{z} \, dz = \hat{z} \times \widehat{\mathbf{T}} \cdot \widehat{\mathbf{T}}' \, dz = \sin^2\theta \, d\phi, \tag{77}$$

and so equation (41) gives

$$2\pi \widetilde{\mathcal{W}}'_{\text{local}}(z) = \frac{1}{(1 + |\cos\theta|)} \sin^2\theta \frac{d\phi}{dz} \tag{78}$$

$$= (1 - |\cos\theta|)\frac{d\phi}{dz}; \tag{79}$$

$$\Rightarrow 2\pi \widetilde{\mathcal{W}}(z_0, z_1) = \int_{z_0}^{z_1} (1 - |\cos\theta|)\frac{d\phi}{dz} \, dz. \tag{80}$$

For $d\phi/dz$ positive and $\theta < \pi/2$, this gives the area swept out between the tantrix and the north pole; if $\theta > \pi/2$ it gives the negative of the area below the tantrix, that is, between the tantrix and the south pole.

6.2. Equivalent closed curves

6.2.1. A curve stretching between two planes. One common technique for defining open writhe consists of extending an open curve to a closed curve, then calculating closed writhe. The result can depend on the extension used; van der Heijden *et al* (2004) use a planar curve between the two endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ of the open curve; this method only works when the tangents at the endpoints are in the same plane as $\mathbf{x}_2 - \mathbf{x}_1$. Starostin (2005) points out that if the tantrix curve is closed by a spherical geodesic, the writhe (as computed by area enclosed on the tantrix sphere) will be well-defined modulo 1. He shows that the axis curve can always be extended in a manner which corresponds to geodesic extensions on the tantrix sphere.

The polar writhe definition does not depend on extending the open curve to a closed curve. It has the advantage that if we chop up a curve into several pieces using parallel slices, then the polar writhes of the slices sum up to the total writhe. However, simple extensions do exist which close a curve in such a way that the polar writhe equals the writhe of the closed curve.

Consider a section of a curve $\mathbf{x}$ for which $ds/dz > 0$. Let the upper endpoint of this section be at height z_1, and the lower endpoint at height z_{-1}. The corresponding tantrix section lies entirely within the northern hemisphere. We can close the tantrix section by connecting its two endpoints to the north pole along longitudinal geodesics.
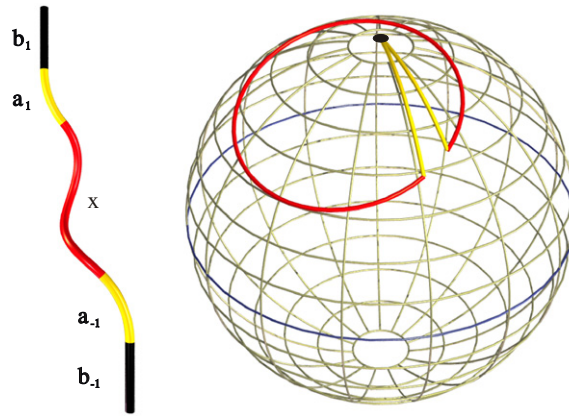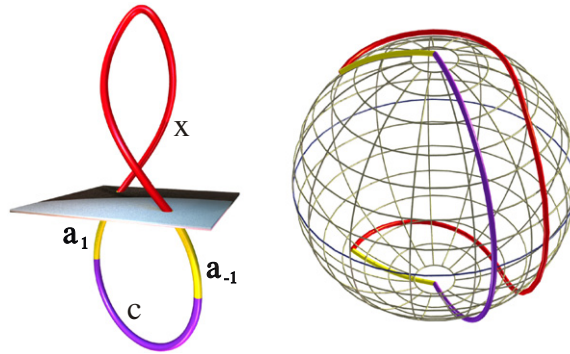
Figure 7. An open curve **x** has been extended by adding circle segments $\mathbf{a}_1$ and $\mathbf{a}_{-1}$, then vertical lines $\mathbf{b}_1$ and $\mathbf{b}_{-1}$. This curve is a helix inclined by a small angle with respect to the vertical (so the tantrix does not centre on the north pole). The helix winds through a phase of 1.9π. For this curve, the polar writhe is $\widetilde{\mathcal{W}} = -0.145$. The associated tantrix curve is also shown.

This corresponds to adding extensions to the axis section **x** (see figure 7). At the upper endpoint $\mathbf{x}(z_1)$ we attach a circular arc $\mathbf{a}_1(z)$ which extends from z_1 to some height $z_2 > 1$. The purpose of this arc is to smoothly align the tangent with the z axis; $\mathbf{a}_1(z)$ has boundary conditions $\mathbf{a}_1(z_1) = \mathbf{x}(z_1)$, $\mathbf{a}_1'(z_1) = \mathbf{x}'(z_1)$ and $\mathbf{a}_1'(z_2) = \hat{z}$. Next we attach a vertical line $\mathbf{b}_1(z)$ to arc $\mathbf{a}_1$ at z_2, i.e. $\mathbf{a}_1(z_2) = \mathbf{b}_1(z_2)$ and $\mathbf{a}_1'(z_2) = \mathbf{b}_1'(z_2) = \hat{z}$. By equation (54) the attachments $\mathbf{a}_1$ and $\mathbf{b}_1$ contribute nothing to the writhe. We extend the lower endpoint z_{-1} with arc $\mathbf{a}_{-1}$ and vertical line $\mathbf{b}_{-1}$ in a similar manner.

The vertical extensions $\mathbf{b}_1$ can extend to $\pm\infty$, so that the curve 'closes at infinity'. Alternatively, we can connect $\mathbf{b}_1$ and $\mathbf{b}_{-1}$ with a planar curve **c** to form a finite closed curve. Here **c** should be chosen to lie in the plane defined by $\hat{z}$ and the relative position vector between the bottom of $\mathbf{b}_1$ and the top of $\mathbf{b}_{-1}$, i.e. $\mathbf{b}_1(z_2) - \mathbf{b}_{-1}(z_{-2})$. Such a curve will have zero local writhe as measured by equation (54), and zero nonlocal writhe (equation (55)) as it does not wind about the rest of the curve.

6.2.2. Loops. By *loop* we mean a curve (figure 8) residing in a half-space, with endpoints on the boundary plane (for example, in the half-space $z > 0$ with endpoints at $z = 0$). This case is most relevant to the astrophysical applications (figure 5). Here a closed curve with equivalent writhe can be constructed in a similar manner to the previous example: first, we join circular sections $\mathbf{a}_1$ and $\mathbf{a}_{-1}$ to the two endpoints. These sections lie below the boundary plane, and have the purpose of smoothly changing the tangent direction to the vertical. The corresponding tantrix curve receives extensions which join the ends to the North and South poles via geodesics. Finally, a semicircle **c** closes the curve.

We have two alternatives: first, we can make the sections $\mathbf{a}_1$ and $\mathbf{a}_{-1}$ arbitrarily small. In this case, they will have negligible nonlocal writhe (i.e. the winding between these two segments becomes arbitrarily small). The local writhes of $\mathbf{a}_1$, $\mathbf{a}_{-1}$ and **c** all vanish. Thus the extended curve will have a closed writhe equal to the polar writhe of **x**. Note that **c** will be in the same plane as the line segment between the two endpoints of **x**. Secondly, we can have $\mathbf{a}_1$ and $\mathbf{a}_{-1}$ of arbitrary (but equal) height. In this case, they may well wind about each other through some angle. Here the extended curve has a different writhe than **x**, but the discrepancy is understood as arising from the polar writhe *below* the boundary plane.

Figure 8. An open curve **x** in the form of a loop with endpoints on the same horizontal plane has been extended by adding circle segments $\mathbf{a}_1$ and $\mathbf{a}_{-1}$, then a semicircle **c**. The associated tantrix curve is also shown. For this curve (see equation (90) with $h = 1.5$ and $\Theta = 3\pi/4$), the polar writhe above the plane is $\widetilde{\mathcal{W}} = 0.55$.

6.3. The writhe of a closed curve

Theorem 6 for a single segment of a curve without turning points is exact. Here we demonstrate the tantrix area theorem, equation (75), calculating up to mod 2. To simplify the discussion, we will only consider a tantrix curve which does not cross itself, and so divides the tantrix sphere into two regions. Tantrices which do cross themselves do not pose any special difficulties; one can simply add up the areas of all subregions (see Aldinger *et al* for a more thorough discussion).

One of the two spherical regions bounded by the tantrix contains the north pole. Let the area of this region be $\mathcal{A}_N$. This is meant to be a signed area: if the tantrix encircles the pole in the westward direction then $\mathcal{A}_N$ will be negative. We wish to show that our expressions for the writhe result in $\mathcal{A}_N$.

For a closed curve, theorems 4 and 5 give

$$\mathcal{W}_r = \frac{1}{2\pi} \sum_{i=1}^{n} \left[\left(\phi\left(z_i^{\min}\right) - \phi\left(z_i^{\max}\right) \right) + \int_{z_i^{\min}}^{z_i^{\max}} (1 - |\cos\theta|) \frac{\mathrm{d}\phi}{\mathrm{d}z} \, \mathrm{d}z \right] - 1 \qquad \text{mod } 2. \qquad (81)$$

Let us look at the contribution of a single piece i beginning and ending at turning points. Each of these pieces has endpoints on the equator. Half of the pieces lie in the northern hemisphere, and half in the south. Let $m = n/2$ count the number of pieces in each hemisphere. We will divide the terms in equation (81) between northern and southern hemispheres; but it will be useful to assign all the tangent angle terms to the southern part (we can do this because the tangent angles are counted twice, e.g. $\phi\left(z_i^{\max}\right) = \phi\left(z_{i+1}^{\max}\right)$ if piece i is in the north):

$$\mathcal{W}_r = (\mathcal{W}_{r\,\text{north}} + \mathcal{W}_{r\,\text{south}}) - 1 \qquad \text{mod } 2; \qquad (82)$$

$$\mathcal{W}_{r\,\text{north}} = \frac{1}{2\pi} \sum_{i=1}^{m} \int_{z_i^{\min}}^{z_i^{\max}} (1 - |\cos\theta|) \frac{\mathrm{d}\phi}{\mathrm{d}z} \, \mathrm{d}z; \qquad (83)$$

$$\mathcal{W}_{r\,\text{south}} = \frac{1}{2\pi} \sum_{i=1}^{m} \left[2\left(\phi\left(z_i^{\min}\right) - \phi\left(z_i^{\max}\right) \right) + \int_{z_i^{\min}}^{z_i^{\max}} (1 - |\cos\theta|) \frac{\mathrm{d}\phi}{\mathrm{d}z} \, \mathrm{d}z \right]. \qquad (84)$$

The northern terms in $\mathcal{W}_{r\,\text{north}}$ give the area swept out between the tantrix and the north pole, as required. In fact, so do the southern terms, but this conclusion requires some

further explanation. For each southern piece, the integral term gives the negative of the area swept out between the tantrix and the south pole. However, the term $2\big(\phi\big(z_i^{\min}\big) - \phi\big(z_i^{\max}\big)\big)$ gives the total area between the longitude lines at $\phi = \phi\big(z_i^{\max}\big)$ and $\phi = \phi\big(z_i^{\min}\big)$. This cancels the negative area of the integral term, leaving the area between the tantrix and the north pole as required. Summing all the northern and southern pieces gives the total area of the northern region $\mathcal{A}_N$, so

$$\mathcal{W}_r = \frac{1}{2\pi}\mathcal{A}_N - 1 \qquad \mathrm{mod}\ 2. \tag{85}$$

6.4. Comparison with Fuller's $\Delta\mathcal{W}_r$ formula

Fuller (1978) showed that the writhe of a closed curve $\mathbf{x}(t)$ could in some circumstances be computed by comparison with a reference curve $\mathbf{x}_{\mathrm{ref}}(t)$. In particular, suppose $\mathbf{x}_{\mathrm{ref}}(t)$ is some given closed curve, which can be smoothly deformed into $\mathbf{x}(t)$. During the deformation, we require that at all times $\widehat{\mathbf{T}}_{\mathrm{ref}}(t)\cdot\widehat{\mathbf{T}}(t) \neq -1$ for all t (t need not be an arclength parametrization). This means that corresponding points on the reference tantrix curve and the deformed tantrix curve are never antipodal. In these circumstances

$$\Delta\mathcal{W}_r = \mathcal{W}_r(\mathbf{x}) - \mathcal{W}_r(\mathbf{x}_{\mathrm{ref}}) = \frac{1}{2\pi}\oint \frac{\widehat{\mathbf{T}}_{\mathrm{ref}}(t) \times \widehat{\mathbf{T}}(t)}{1 + \widehat{\mathbf{T}}_{\mathrm{ref}}(t)\cdot\widehat{\mathbf{T}}(t)} \cdot (\widehat{\mathbf{T}}'_{\mathrm{ref}}(t) + \widehat{\mathbf{T}}'(t))\,\mathrm{d}t. \tag{86}$$

A rigorous proof can be found in Aldinger *et al* (1995). Cantarella (2005) shows that this formula gives the area of the ribbon on the tantrix sphere between the two tantrix curves, and extends the formula to polygonal curves.

For our purposes we wish to investigate how the formula works when $\mathbf{x}_{\mathrm{ref}}$ is, in essence, a vertical line. More explicitly, suppose that $\mathbf{x}_{\mathrm{ref}}$ is a very large circle,

$$\mathbf{x}_{\mathrm{ref}}(t) = R(\cos t, 0, \sin t), \qquad R \gg 1. \tag{87}$$

Then near $t = 0$ the reference curve is to a good approximation vertical. The writhe of a stretch of curve $\mathbf{x}$ of length order unity could then be compared to this vertical region. The formula, with $\widehat{\mathbf{T}}_{\mathrm{ref}} = \hat{z}$, gives

$$\Delta\mathcal{W}_r = \frac{1}{2\pi}\int \frac{\hat{z} \times \widehat{\mathbf{T}}_{\mathrm{ref}}(t)}{1 + T_z(t)} \cdot \widehat{\mathbf{T}}'(t))\,\mathrm{d}t \tag{88}$$

$$= \frac{1}{2\pi}\int \frac{1}{1 + \lambda(t)}(\widehat{\mathbf{T}}(t) \times \widehat{\mathbf{T}}'(t))_z\,\mathrm{d}t \tag{89}$$

where $\lambda = \cos\theta = \widehat{\mathbf{T}}_z$. If we assume the reference curve has zero writhe, this gives the writhe of $\mathbf{x}$.

This result looks superficially similar to our expression for the local writhe, equation (54) from theorem 4. There are two differences: first, equation (89) involves λ rather than its absolute value $|\lambda|$. Secondly, theorem 4 adds an extra term, the nonlocal writhe equation (55), to the total writhe. What is happening? Consider a small section of the tantrix curve from t to $t + \mathrm{d}t$. If it is in the northern hemisphere, the integrands in both equations (54) and (89) are identical: they give the spherical area between the tantrix section and the north pole. But if the tantrix section is in the southern hemisphere, the behaviour is quite different. The local writhe equation (54) gives area below the curve down to the south pole, while equation (89) still gives area above up to the north pole.

The differences become most apparent if we consider almost vertical sections of a curve with small wiggles. For upward travelling curves the tantrix will be close to the north pole and

hence sweep out a small area in between. Both equations (54) and (89) give a small number as expected. But what happens when a section of curve travels almost vertically downward, again with small wiggles? The local writhe equation (54) still reports a small contribution, because there is little area between the tantrix section and the south pole. The $\Delta\mathcal{W}_r$ formula equation (89), however, assigns such a section a large amount of writhe, as it is measuring area all the way up to the north pole. Maggs (2001) and Rossetto and Maggs (2003) have shown that statistical distributions of writhe (as measured by $\Delta\mathcal{W}_r$) for curves generated by random walks can be strongly affected by fluctuations near the south pole.

The discrepancy between the two formulae lies in the treatment of nonlocal contributions to the writhe. In order for our stretch of curve **x** to be reachable from the reference curve by smooth deformations (never anti-podal) we must have **x** travelling upwards at the top and the bottom of its range. It must then have turning points if part of it goes downwards. The different sections which loop up and down can then wind about each other, contributing to the nonlocal writhe term equation (55).

Thus, there are several advantages to employing the polar writhe formulae (54) and (55) rather than the $\Delta\mathcal{W}_r$ formula (89). First, there is no requirement of smooth never anti-podal deformation from a reference curve. Secondly, the local writhe formula is more balanced: upward and downward travelling curves (with corresponding northern and southern tantrices) have integrands identical in magnitude, because of the absolute value signs. This is an especially desirable property because total writhe does not change on reversal $s \to -s$. Third, the behaviour of equation (89) near the south pole can magnify experimental and numerical errors, and affect statistical analyses (Maggs 2001): the integral is heavily weighted toward small southern wiggles. Fourth, the polar writhe explicitly displays the influence of nonlocal windings.

7. Examples

7.1. Twisted parabolas

Consider curves with two endpoints on one plane as in sections 5.2 and 6.2.2. Let

$$z(t) = 4ht(1-t) \qquad 0 \leqslant t \leqslant 1;$$
$$f(t) = \left(\left(t - \frac{1}{2} \right) \cos \frac{\Theta z(t)}{h}, \left(t - \frac{1}{2} \right) \sin \frac{\Theta z(t)}{h}, z(t) \right), \tag{90}$$

where h gives the maximum height of the curve $f(t)$ above $z = 0$, and Θ gives the maximum amount that the loop has been twisted (see figures 8, 9 and 10). Note that $\widetilde{\mathcal{W}}_{\text{nonlocal}} = -\Theta/\pi$. The local writhe is of the opposite sign to the nonlocal writhe, therefore decreasing the magnitude of the total. As height increases, the influence of the local writhe becomes less important. Curiously, at a particular height $h \approx 0.3734$ the writhe vanishes, $\widetilde{\mathcal{W}} = 0$, for all values of Θ.

Figure 9 to the lower right shows a picture of a twisted parabola as seen from above, with $\Theta = -\pi$. Note that if we follow the curve from either end to the middle, the curve travels both upwards and clockwise, resulting in a negative local writhe (whereas this curve has a positive nonlocal writhe).
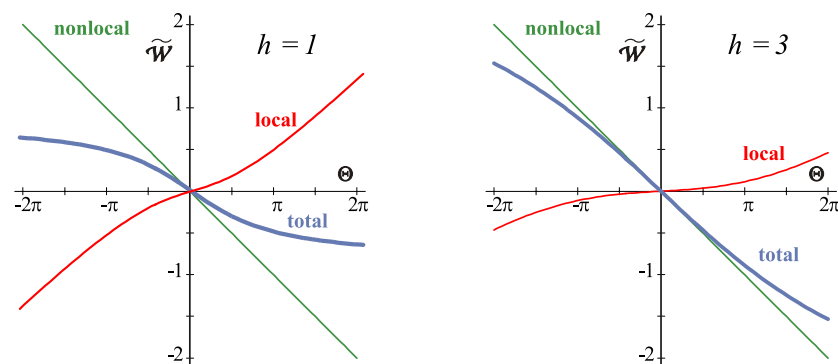
7.2. Helices

Here we consider the effect of rotation of the central axis on a helix

$$g(t) = \left(\sin(2\pi t), \cos(2\pi t), 2\pi \left(t - \tfrac{1}{2} \right) \right) \qquad 0 \leqslant t \leqslant 1. \tag{91}$$
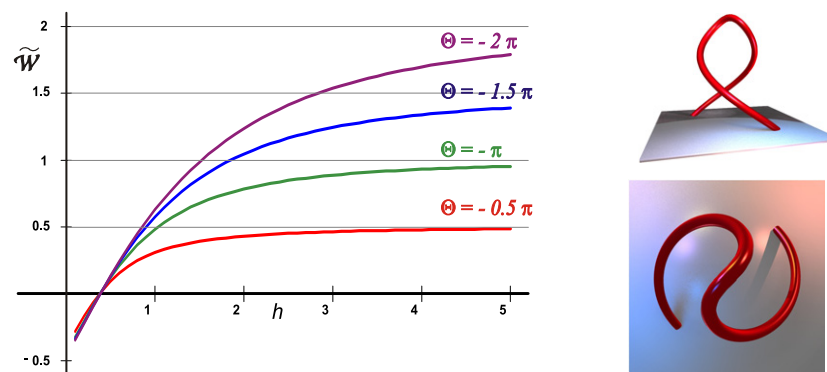
Figure 9. Writhe $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(0, \infty)$ calculated for the twisted parabolas of section 7.1. The curve displayed to the right has $\Theta = -\pi$ and $h = 1$, seen from the side and seen from above.

Figure 10. Twisted parabolas: $\widetilde{\mathcal{W}}_{\text{local}}$, $\widetilde{\mathcal{W}}_{\text{nonlocal}}$ and total polar writhe $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(0, \infty)$ for heights $h = 1$ and $h = 3$.

The helix is rotated through an angle ψ about the x axis (see figure 11). For $\psi < \pi/4$ there are no turning points in z, so no nonlocal writhe. Between $\psi = \pi/4$ and approximately $\psi = 0.301\pi$, the endpoint at $t = 1$ still has the maximum z value, i.e. $z(1) = z^{\text{max}}$ (also the $t = 0$ endpoint has minimum height $z(0) = z^{\text{min}}$). Thus up to 0.3π the curve simply stretches between the planes $z = z^{\text{min}}$ and $z = z^{\text{max}}$, with endpoints on these planes. This is the situation discussed in section 6.2.1. The tantrix curve closes upon itself, as the helix completes one period of 2π. See figure 7 for a similar curve; for illustrative purposes this curve only makes it to 1.9π. The writhe then equals the area inside the tantrix. As ψ increases to 0.3π, the helix leans more and more, and the tantrix moves southwards, but the writhe stays constant at $\widetilde{\mathcal{W}} = -0.29$.

However, above $\psi = 0.301\pi$ the $t = 1$ endpoint dips below the maximum point on the curve. We no longer have the simple situation of a curve stretching between parallel planes, and there are extra nonlocal writhe terms between the last part of the curve (between z^{max} and the end) and the rest of the curve. As a consequence the polar writhe deviates away from -0.29. This illustrates how the polar writhe may not be appropriate when endpoints lie within the range in z, i.e. if $z^{\text{min}} < z(0) < z^{\text{max}}$ or $z^{\text{min}} < z(1) < z^{\text{max}}$.
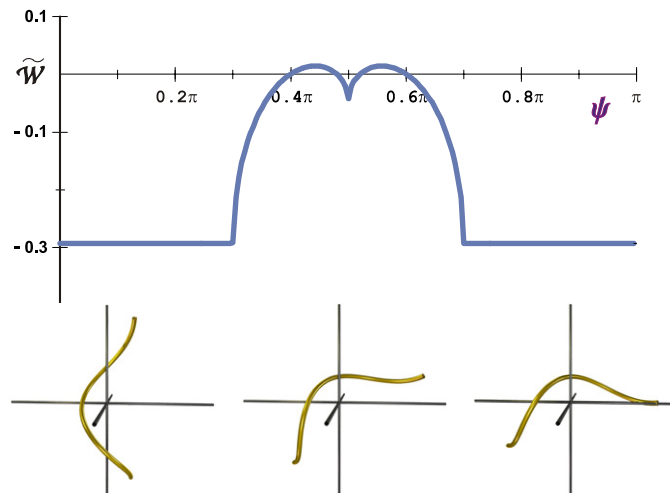
Figure 11. Let **x** be a helical curve with a net turn of 2π is aligned along a central axis inclined by an angle ψ with respect to the vertical. The polar writhe $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(-\infty, \infty) = \widetilde{\mathcal{W}}(-z^{\min}, z^{\max})$ is shown as a function of ψ. Below is shown the curve for $\psi = 0$, $\psi = 0.3\pi$ and $\psi = 0.4\pi$ (equation (91)).

Acknowledgments

Appendix. Proofs of theorems

Proof of theorem 1. We will prove the theorem by counting crossings. Thus we employ equation (10) to relate the linking number to the signed number of crossings as seen with some projection angle. We will be particularly interested in projections perpendicular to $\hat{\mathbf{z}}$, i.e. with $\hat{\mathbf{n}}(\psi) = \cos\psi\,\hat{x} + \sin\psi\,\hat{y}$ for some azimuthal direction ψ (see Berger 1993 for a similar procedure). Let $C(\psi)$ be the number of crossings seen from direction ψ. The *average* signed number of crossings seen from these side directions is

$$\bar{f} = \frac{1}{2\pi} \oint_0^{2\pi} C(\psi)\,\mathrm{d}\psi. \tag{A.1}$$

Now consider piece i of curve **x** and piece j of curve **y**. Suppose for the moment that both of these curve segments point upwards, $\mathrm{d}s/\mathrm{d}z > 0$. Suppose also that their extent in the z direction overlaps between $z = z_1$ and $z = z_2$. Then in this interval $\sigma_i = \sigma_j = 1$, and they wrap around each other through a net angle

$$\Delta\Theta_{ij} = \int_{z_1}^{z_2} \frac{\mathrm{d}\Theta_{ij}}{\mathrm{d}z}\,\mathrm{d}z. \tag{A.2}$$

In other words, the relative position vector $\mathbf{r}_{ij}$ rotates through a net angle $\Delta\Theta_{ij}$ between z_1 and z_2.

We assert that for pieces i and j, the perpendicular crossing number is

$$\bar{f}_{ij} = \sigma_i\sigma_j\frac{\Delta\Theta_{ij}}{\pi}. \tag{A.3}$$

To demonstrate this, consider an observer at azimuthal angle ϕ. This observer will see a crossing at heights z where the relative position vector $\mathbf{r}_{ij}$ points in the $\pm\phi$ direction. Now $\mathbf{r}_{ij}$ may rotate as it travels from z_1 to z_2. If $\mathbf{r}_{ij}$ swings all the way around n times between z_1 and z_2 ($\Delta\Theta_{ij} = 2\pi n$) then each observer will see $2n$ crossings (n times for when the vector points toward the observer, and n times for when the vector points away from the observer). Thus the quantity $\Delta\Theta_{ij}$ relates to how many times each observer sees a crossing.

If $|\Delta\Theta_{ij}| < \pi$ some observers will not see a crossing; in this case, $|\Delta\Theta_{ij}|/\pi$ gives the fraction of observers seeing a crossing. Note that $\mathbf{r}_{ij}$ may wiggle back and forth, i.e. $\mathrm{d}\Theta_{ij}/\mathrm{d}z$ may not stay the same sign. But in this case observers will see crossings of both signs, which cancel out. Thus $\Delta\Theta_{ij}/\pi$ gives the *net* number of crossings, averaged over all projection angles, i.e. $\bar{f}_{ij}$.

So far we have assumed that $\sigma_i = \sigma_j = 1$. In general, the sign of the crossings will be positive if $\sigma_i = \sigma_j$ and $\Delta\Theta_{ij} > 0$. The sign becomes negative if one of the σs changes sign. Thus $\Delta\Theta_{ij}$ must be weighted by the product $\sigma_i\sigma_j$, leading to equation (A.3).

We now sum over all pairs of curve segments to give

$$\bar{f} = \frac{1}{\pi} \sum_{i=1}^{N} \sum_{j=1}^{M} \sigma_i \sigma_j \Delta\Theta_{ij}. \tag{A.4}$$

Finally, for a closed link, the signed number of crossings is the same for all projection angles and equals $2\mathcal{L}_k$. Thus the average $\bar{f}$ will also have this value,

$$\bar{f} = 2\mathcal{L}_k, \tag{A.5}$$

thus proving the theorem.

Proof of theorem 3. We surround the axis curve $\mathbf{x}$ with a tube. As in section 2, there is a family of secondary curves on the surface of the tube labelled by β; the $\beta = 0$ curve follows $\mathbf{x} + \epsilon\widehat{\mathbf{V}}$. We will calculate $\widetilde{\mathcal{L}}'$ and $\widetilde{\mathcal{T}}'$ for the secondary curves in the tube surface. The next step will be to average $\widetilde{\mathcal{L}}' - \widetilde{\mathcal{T}}'$ over all the secondary curves, and show that this average only depends on the geometry of the axis.

Let $\widehat{\mathbf{W}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{V}}$ so that $\{\widehat{\mathbf{T}}, \widehat{\mathbf{V}}, \widehat{\mathbf{W}}\}$ is an orthonormal right-handed frame. We consider the neighbourhood of some point on the curve which is not a maxima or minima, so that we can parametrize the curve by z. As twist and writhe do not change under a reversal of the direction of the curve, we can assume $\lambda = \mathrm{d}z/\mathrm{d}s > 0$, i.e. $\sigma > 0$.

The secondary curve labelled by β passes through the points $\mathbf{y}(z, \beta) = \mathbf{x}(z) + \epsilon\widehat{\mathbf{U}}(z, \beta)$ where

$$\widehat{\mathbf{U}}(z, \beta) = \cos\beta\widehat{\mathbf{V}}(z) + \sin\beta\widehat{\mathbf{W}}(z). \tag{A.6}$$

Note that $\mathrm{d}(\widehat{\mathbf{V}} \cdot \widehat{\mathbf{V}})/\mathrm{d}z = 0$, so $\widehat{\mathbf{V}} \cdot \widehat{\mathbf{V}}' = 0$ (we will no longer write the (z) dependence everywhere). Similarly, $\widehat{\mathbf{W}} \cdot \widehat{\mathbf{W}}' = 0$. Also, as $\widehat{\mathbf{V}} \cdot \widehat{\mathbf{W}} = 0$,

$$\omega \equiv \widehat{\mathbf{V}}' \cdot \widehat{\mathbf{W}} = -\widehat{\mathbf{V}} \cdot \widehat{\mathbf{W}}'. \tag{A.7}$$

These relations simplify the expression for the twist of the β curve, $\widetilde{\mathcal{T}}'(z, \beta)$. From equation (30) with $\sigma > 0$,

$$2\pi\widetilde{\mathcal{T}}' = \widehat{\mathbf{T}} \cdot \widehat{\mathbf{U}} \times \widehat{\mathbf{U}}' \tag{A.8}$$

$$= \widehat{\mathbf{T}} \times (\cos\beta\widehat{\mathbf{V}} + \sin\beta\widehat{\mathbf{W}}) \cdot (\cos\beta\widehat{\mathbf{V}}' + \sin\beta\widehat{\mathbf{W}}') \tag{A.9}$$

$$= (\cos\beta\widehat{\mathbf{W}} - \sin\beta\widehat{\mathbf{V}}) \cdot (\cos\beta\widehat{\mathbf{V}}' + \sin\beta\widehat{\mathbf{W}}') \tag{A.10}$$

$$= \omega. \tag{A.11}$$

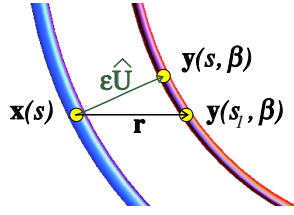Note that the twist $\widetilde{\mathcal{T}}'$ is independent of β.

Figure A.1. The points $\mathbf{x}(z) = \mathbf{x}(s)$, $\mathbf{y}(s, \beta) = \mathbf{x}(s) + \epsilon\widehat{\mathbf{U}}$ and $\mathbf{y}(s_1, \beta) = \mathbf{x}(s) + \mathbf{r}$.

Next consider $\widetilde{\mathcal{L}}'$ using equation (19), as well as equation (25) applied to just the single pair $\mathbf{x}$ and $\mathbf{y}(\beta)$:

$$2\pi\widetilde{\mathcal{L}}' = \frac{\mathrm{d}\Theta(\mathbf{x}, \mathbf{y})}{\mathrm{d}z} = \frac{\hat{\mathbf{z}} \cdot \mathbf{r}(z) \times \mathbf{r}'(z)}{|\mathbf{r}(z)|^2}. \tag{A.12}$$

Here $\mathbf{r}$ points from $\mathbf{x}(z)$ to the point on the secondary curve *at the same height z* (see figure A.1). Let the arclength along the axis at the point $\mathbf{x}(z)$ be s. The tip of the $\mathbf{r}$ arrow is a point P on the secondary corresponding to a different axis arclength s_1:

$$\mathbf{y}(s_1, \beta) = \mathbf{x}(s_1) + \epsilon\widehat{\mathbf{U}}(s_1, \beta) \tag{A.13}$$

$$\approx \mathbf{x}(s) + \epsilon\widehat{\mathbf{U}}(s, \beta) + \left(\widehat{\mathbf{T}}(s) + \epsilon\frac{\mathrm{d}\widehat{\mathbf{U}}(s, \beta)}{\mathrm{d}s}\right)(s_1 - s). \tag{A.14}$$

To first order in ϵ,

$$\mathbf{r} = \mathbf{y}(s_1, \beta) - \mathbf{x}(s) \tag{A.15}$$

$$\approx \epsilon\widehat{\mathbf{U}}(s, \beta) + \widehat{\mathbf{T}}(s)(s_1 - s). \tag{A.16}$$

By definition $r_z = 0$, so

$$s_1 - s \approx -\epsilon U_z(s)/\hat{T}_z(s) = -\epsilon U_z(s)/\lambda(s). \tag{A.17}$$

Thus to first order in ϵ

$$\mathbf{r} = \epsilon(\widehat{\mathbf{U}} - \lambda^{-1}U_z\widehat{\mathbf{T}}). \tag{A.18}$$

We define $\mathbf{R} = \mathbf{r}/\epsilon$. Then (in the limit $\epsilon \to 0$)

$$2\pi\widetilde{\mathcal{L}}' = \frac{\hat{\mathbf{z}} \cdot \mathbf{R} \times \mathbf{R}'}{|\mathbf{R}|^2}, \qquad \mathbf{R} = \widehat{\mathbf{U}} - \lambda^{-1}U_z\widehat{\mathbf{T}}. \tag{A.19}$$

To go further, we will need two new orthonormal frames, and decompose $\mathbf{R}$ in these frames. Let $\mu = |\hat{\mathbf{z}} \times \widehat{\mathbf{T}}|$. The first new frame will be

$$\{\widehat{\mathbf{T}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\} = \{\widehat{\mathbf{T}}, \hat{\mathbf{z}} \times \widehat{\mathbf{T}}/\mu, \widehat{\mathbf{T}} \times (\hat{\mathbf{z}} \times \widehat{\mathbf{T}}/\mu)\}. \tag{A.20}$$

(The case where $\widehat{\mathbf{T}}$ is parallel to $\hat{\mathbf{z}}$ will be discussed at the end of the proof.) As $\widehat{\mathbf{V}}$ and $\widehat{\mathbf{W}}$ are perpendicular to $\widehat{\mathbf{T}}$, we can write

$$\begin{pmatrix} \widehat{\mathbf{V}} \\ \widehat{\mathbf{W}} \end{pmatrix} = \begin{pmatrix} \cos\psi(z) & \sin\psi(z) \\ -\sin\psi(z) & \cos\psi(z) \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}} \\ \widehat{\mathbf{g}} \end{pmatrix} \tag{A.21}$$

for some angle $\psi(z)$. Then from equation (A.6),

$$\widehat{\mathbf{U}} = \cos(\beta + \psi)\widehat{\mathbf{f}} + \sin(\beta + \psi)\widehat{\mathbf{g}}. \tag{A.22}$$

Next let

$$\widehat{\mathbf{h}} = \hat{\mathbf{z}} \times \widehat{\mathbf{f}} = -\widehat{\mathbf{T}}_\perp/\mu, \tag{A.23}$$

and go to the frame $\{\hat{\mathbf{z}}, \widehat{\mathbf{f}}, \widehat{\mathbf{h}}\}$. In terms of these vectors

$$\widehat{\mathbf{g}} = \mu^{-1}(\hat{\mathbf{z}} - T_z\widehat{\mathbf{T}}) \tag{A.24}$$

$$= \mu^{-1}((1 - \lambda^2)\hat{\mathbf{z}} - \lambda T_\perp) \tag{A.25}$$

$$= \mu\hat{\mathbf{z}} + \lambda\widehat{\mathbf{h}}, \tag{A.26}$$

$$\widehat{\mathbf{T}} = \lambda\hat{\mathbf{z}} - \mu\widehat{\mathbf{h}}. \tag{A.27}$$

Substituting for $\widehat{\mathbf{g}}$ in equation (A.22) gives

$$\widehat{\mathbf{U}} = \mu \sin(\beta + \psi)\hat{\mathbf{z}} + \cos(\beta + \psi)\widehat{\mathbf{f}} + \lambda \sin(\beta + \psi)\widehat{\mathbf{h}}. \tag{A.28}$$

Finally, from equation (A.19)

$$\mathbf{R} = \cos(\beta + \psi)\widehat{\mathbf{f}} + (\lambda + \lambda^{-1}\mu^2) \sin(\beta + \psi)\widehat{\mathbf{h}} \tag{A.29}$$

$$= \cos(\beta + \psi)\widehat{\mathbf{f}} + \lambda^{-1} \sin(\beta + \psi)\widehat{\mathbf{h}}. \tag{A.30}$$

and

$$\mathbf{R}^2 = (\cos^2(\beta + \psi) + \lambda^{-2} \sin^2(\beta + \psi)). \tag{A.31}$$

The z derivative is

$$\mathbf{R}' = \cos(\beta + \psi)(\lambda^{-1}\psi'\widehat{\mathbf{h}} + \widehat{\mathbf{f}}') + \sin(\beta + \psi)(-\psi'\widehat{\mathbf{f}} + \lambda^{-1}\widehat{\mathbf{h}}' - \lambda'\lambda^{-2}\widehat{\mathbf{h}}). \tag{A.32}$$

We now proceed to calculate equation (A.19). Simple vector identities give

$$(\widehat{\mathbf{f}} \times \widehat{\mathbf{h}}' + \widehat{\mathbf{h}} \times \widehat{\mathbf{f}}') \cdot \hat{\mathbf{z}} = 0, \tag{A.33}$$

which helpfully removes a few terms. Also,

$$\widehat{\mathbf{f}} \times \widehat{\mathbf{f}}' = \widehat{\mathbf{h}} \times \widehat{\mathbf{h}}' = \mu^{-2}\hat{\mathbf{z}} \cdot \widehat{\mathbf{T}} \times \widehat{\mathbf{T}}' \tag{A.34}$$

$$= \mu^{-2}\lambda^{-1}\kappa B_z. \tag{A.35}$$

Combining equations (A.30) to (A.35) gives

$$\frac{\hat{\mathbf{z}} \cdot \mathbf{R} \times \mathbf{R}'}{\mathbf{R}^2} = \frac{\lambda\psi' - \lambda' \cos(\beta + \psi) \sin(\beta + \psi)}{(\lambda^2 \cos^2(\beta + \psi) + \sin^2(\beta + \psi))} + \frac{\kappa}{\lambda\mu^2}B_z. \tag{A.36}$$

Suppose we now average this expression over all secondary curves in the tube, i.e., over $0 \leqslant \beta < 2\pi$. The term involving λ' vanishes, and the last term is unaffected. The first term gives

$$\frac{\lambda\psi'}{2\pi} \int_0^{2\pi} \frac{1}{(\lambda^2 \cos^2(\beta + \psi) + \sin^2(\beta + \psi))} \, \mathrm{d}\beta = \psi'. \tag{A.37}$$

Thus equation (A.19) finally gives

$$2\pi \widetilde{\mathcal{L}}' = \psi' + \frac{1}{\lambda\mu^2}\kappa B_z. \tag{A.38}$$

Meanwhile, from equation (A.11),

$$2\pi \widetilde{T}' = \omega = \widehat{\mathbf{V}}' \cdot \widehat{\mathbf{W}} \tag{A.39}$$

$$= \psi' + (\cos\psi \widehat{\mathbf{f}}' + \sin\psi \widehat{\mathbf{g}}') \cdot (-\sin\psi \widehat{\mathbf{f}} + \cos\psi \widehat{\mathbf{g}}). \tag{A.40}$$

Now the orthonormal vectors satisfy $\widehat{\mathbf{f}} \cdot \widehat{\mathbf{f}}' = \widehat{\mathbf{g}} \cdot \widehat{\mathbf{g}}' = 0$, while $\widehat{\mathbf{f}} \cdot \widehat{\mathbf{g}}' = -\widehat{\mathbf{f}}' \cdot \widehat{\mathbf{g}}$, so

$$2\pi \widetilde{T}' = (\psi' + \widehat{\mathbf{f}}' \cdot \widehat{\mathbf{g}}) = (\psi' + \mu^{-2}\widehat{\mathbf{z}} \times \widehat{\mathbf{T}}' \cdot (\widehat{\mathbf{T}} \times (\widehat{\mathbf{z}} \times \widehat{\mathbf{T}}))) \tag{A.41}$$

$$= \psi' + \frac{\lambda}{\mu^2} \widehat{\mathbf{z}} \cdot \widehat{\mathbf{T}} \times \widehat{\mathbf{T}}' \tag{A.42}$$

$$= \psi' + \frac{1}{\mu^2} \kappa B_z. \tag{A.43}$$

Thus

$$2\pi \widetilde{\mathcal{W}}' = \frac{(1-\lambda)}{\mu^2} \frac{\kappa B_z}{\lambda}. \tag{A.44}$$

This proves the theorem for $0 < \lambda < 1$. For vertical points on the axis curve ($\lambda = 1$) the expression for $\widetilde{\mathcal{W}}'$ gives 0. This is expected, because for such points the rate of change of linking $\widetilde{\mathcal{L}}'$ should coincide with the rate of change of twisting $\widetilde{T}'$ (the first measures winding about $\widehat{\mathbf{z}}$, while the second measures winding about $\widehat{\mathbf{T}}$, and for vertical points $\widehat{\mathbf{z}} = \widehat{\mathbf{T}}$). Thus the theorem extends to vertical points.

Finally, if the axis parameter s is reversed, then $\lambda \to -\lambda$ and $B_z \to -B_z$, but $\widetilde{\mathcal{W}}'$ should not change. In this case

$$2\pi \widetilde{\mathcal{W}}' = \frac{(1-|\lambda|)}{\mu^2} \frac{\kappa B_z}{\lambda} = \frac{1}{2\pi} \frac{1}{(1+|\lambda_i|)} \frac{\kappa B_z}{\lambda}. \tag{A.45}$$

Proof of theorem 5. By equation (55),

$$\widetilde{\mathcal{W}}'_{\text{nonlocal}} = \frac{1}{2\pi} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \Theta'_{ij}(z). \tag{A.46}$$

Consider the double sum of nonlocal terms. Some of the terms may vanish because pieces i and j may not exist at the same height z (i.e. $\sigma_i \sigma_j = 0$ everywhere). Suppose two pieces i and j do overlap in z, from some height $z_{ij}^{\min}$ up to some height $z_{ij}^{\max}$. When we integrate over this overlap, we obtain the difference between two angles. Let $\Theta_{ij}^{\min}$ and $\Theta_{ij}^{\max}$ be the orientations of the relative position vectors $\mathbf{r}_{ij}(z_{ij}^{\min})$ and $\mathbf{r}_{ij}(z_{ij}^{\max})$ between the points $\mathbf{x}_i$ and $\mathbf{x}_j$ on the curves at these two heights. Note that $i < j$ and $\mathbf{r}_{ij}$ points from piece i to piece j. Also note that for adjoining pieces, one of the angles will be the orientation of the tangent vector at the join. (For example, in figure 4, $\Theta_{12}^{\min} = \Theta_{PC}$ and $\Theta_{12}^{\max} = \phi_B$.)

With this notation,

$$\widetilde{\mathcal{W}}_{ij} = \frac{\sigma_i \sigma_j}{2\pi} \int_{z_{ij}^{\min}}^{z_{ij}^{\max}} \Theta'_{ij}(z) \, dz = \frac{\sigma_i \sigma_j}{\pi} \left(\Theta_{ij}^{\max} - \Theta_{ij}^{\min} \right) + w_{ij}. \tag{A.47}$$

Suppose $\Theta_{ij}^{\max}$ is not a tangent vector. At least one of the points $\mathbf{x}_i^{\max}$ or $\mathbf{x}_j^{\max}$ is at a local maximum in z, say $\mathbf{x}_i^{\max}$. This point joins piece i with either piece $i - 1$ or $i + 1$. Suppose it is $i + 1$. Consider $\widetilde{\mathcal{W}}_{(i+1)j}$. From the previous equation, this will involve the angle $\Theta_{(i+1)j}^{\max}$.

This angle is measured with the same point $\mathbf{x}_i = \mathbf{x}_{i+1}$, so $\Theta^{\max}_{(i+1)j} = \Theta^{\max}_{ij}$. But $\sigma_{i+1} = -\sigma_i$; consequently

$$\sigma_{i+1}\sigma_j\Theta^{\max}_{(i+1)j} = -\sigma_i\sigma_j\Theta^{\max}_{ij} \tag{A.48}$$

and the two terms cancel.

The same would hold true if the maximum were between $i-1$ and i, or if it involved j instead of i. Also, the cancellation of Θ terms occurs at minima as well. As a result, all of the angles cancel except for the tangent vectors connecting adjoining pieces. (These do not cancel because they only appear once.)

Now, for adjoining pieces $\sigma_i\sigma_{i+1} = -1$. Let $\alpha_i = -1$ if the end of piece i is a maximum and $\alpha_i = +1$ if it is a minimum. Also the angles Θ become the orientations ϕ of tangent vectors. More precisely, except for the point joining piece 1 with the last piece $i = 2m = n$, we can write $\Theta_{i(i+1)} = \phi_{i(i+1)}$. Note that the point joining piece 1 with piece m has $\mathbf{r}_{1n}$ reversed with respect to the tangent vector. Thus $\Theta_{1n} = \phi_{n1} \pm \pi$.

Thus the nonlocal terms (including the winding numbers w_{ij}) sum to

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \frac{1}{\pi}\sum_{i=1}^{n-1}\alpha_i\Theta_{i(i+1)} + \frac{1}{\pi}\alpha_n\Theta_{1n} + 2w \tag{A.49}$$

$$= \frac{1}{\pi}\sum_{i=1}^{n-1}\alpha_i\phi_{i(i+1)} + \frac{1}{\pi}\alpha_n(\phi_{n1} \pm \pi) + 2w. \tag{A.50}$$

Now at minima $\alpha_i\phi_{i(i+1)} = \phi\left(z_i^{\min}\right) = \phi\left(z_{i+1}^{\min}\right)$ and at maxima $\alpha_i\phi_{i(i+1)} = -\phi\left(z_i^{\max}\right) = -\phi\left(z_{i+1}^{\max}\right)$. Taking into account the double counting,

$$\widetilde{\mathcal{W}}_{\text{nonlocal}} = \frac{1}{2\pi}\sum_{i=1}^{n}\left(\phi\left(z_i^{\min}\right) - \phi\left(z_i^{\max}\right)\right) + (2w \pm 1). \tag{A.51}$$

Calculating mod 2 completes the theorem.

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